

Available online at www.sciencedirect.com

SCIENCE @ DIRECT®

Journal of Algebra 284 (2005) 679–704

JOURNAL OF
Algebrawww.elsevier.com/locate/jalgebra

Weak C -cleft extensions, weak entwining structures and weak Hopf algebras

J.N. Alonso Álvarez ^a, J.M. Fernández Vilaboa ^b,
R. González Rodríguez ^{c,*}, A.B. Rodríguez Raposo ^b

^a *Departamento de Matemáticas, Universidad de Vigo, Lagoas-Marcosende, Vigo, E-36280, Spain*

^b *Departamento de Álgebra, Universidad de Santiago de Compostela, Santiago de Compostela, E-15771, Spain*

^c *Departamento de Matemática Aplicada II, Universidad de Vigo, Lagoas-Marcosende, Vigo, E-36280, Spain*

Received 10 May 2004

Available online 23 December 2004

Communicated by Eva Bayer-Fluckiger

Abstract

We formulate the concept of weak cleft extension for a weak entwining structure in a braided monoidal category \mathcal{C} with equalizers and coequalizers. We prove that if A is a weak C -cleft extension, then there is an isomorphism of algebras between A and a subobject of the tensor product of A_C and C where A_C is a subalgebra of A . Also, we prove the corresponding dual results and linking the information of this two parts we obtain a general property for a pair morphisms $f : C \rightarrow A$ and $g : A \rightarrow C$ of algebras and coalgebras satisfying certain conditions. Finally, as particular instances, we get the results of Fernández and Rodríguez, the theorems of Radford, Majid and Bespalov (in the case of Hopf algebras with projection) and the ones obtained by Alonso and González for weak Hopf algebras living in a symmetric category with split idempotents, for example, the weak theorem of Blattner, Cohen and Montgomery for weak Hopf algebras with coalgebra splitting is one of them. © 2004 Elsevier Inc. All rights reserved.

Keywords: Weak entwining structure; Weak C -cleft extension; Weak Hopf algebra

* Corresponding author.

E-mail addresses: jnalonso@uvigo.es (J.N. Alonso Álvarez), alvila@usc.es (J.M. Fernández Vilaboa), rgon@dma.uvigo.es (R. González Rodríguez), abraposo@usc.es (A.B. Rodríguez Raposo).

Introduction

Weak entwining structures have been introduced by Caenepeel and de Groot [16] as a generalization of entwining structures defined by Brzeziński and Majid [14,15]. They introduce the so-called entwining structures, consisting of an algebra A , a coalgebra C , and an intertwining $\psi : C \otimes A \rightarrow A \otimes C$ satisfying four technical conditions which have been replaced for weaker axioms in the definition of Caenepeel and de Groot. In this context, a weak entwined module is at the same time an A -module a C -comodule, with compatibility condition given by ψ . With this definition it is possible to unify some categories of modules associated to a Hopf algebra or a weak Hopf algebra, for example classical Doi–Hopf modules and weak Doi–Hopf modules defined by Böhm in [10].

On the other hand, the main result of [3] is a generalization, for weak Hopf algebras living in a symmetric monoidal category with split idempotents, of the well-known result due to Blattner, Cohen and Montgomery which shows that if $g : B \rightarrow H$ is a morphism of Hopf algebras with coalgebra splitting f , then there exists an algebra isomorphism between B and the crossed product $B_H \sharp_{\sigma_{B_H}} H$ where B_H is the left Hopf kernel of g and σ_{B_H} is a suitable cocycle (see Theorem (4.14) of [9]). In this generalization Alonso and González proved that if $g : B \rightarrow H$ is a morphism of weak Hopf algebras and there exists a morphism of coalgebras $f : H \rightarrow B$ such that $g \circ f = id_H$ and $f \circ \eta_H = \eta_B$, then using the idempotent morphism $q_H^B = \mu_B \circ (B \otimes (\lambda_B \circ f \circ g)) \circ \delta_B : B \rightarrow B$ and an equalizer diagram it is possible to construct an algebra B_H and morphisms $\varphi_{B_H} : H \otimes B_H \rightarrow B_H$, $\sigma_{B_H} : H \otimes H \rightarrow B_H$ such that there exists a subobject $B_H \times H$ of $B_H \otimes H$ isomorphic with B as algebras and with algebra structure defined by a crossed product involving φ_{B_H} and σ_{B_H} . Also, in [3] one can find the dual results and linking this information with the one obtained previously, we get a weak version of Radford’s theorem introducing the category of weak Yetter–Drinfeld modules. Radford’s theorem [27] gives equivalent conditions for an object $A \otimes H$ equipped with smash product algebra and coalgebra to be a Hopf algebra and characterizes such objects via a bialgebra projection. Majid in [24] interpreted this result in the modern context of Yetter–Drinfeld modules and stated that there is a one to one correspondence between Hopf algebras in this category, denoted by ${}^H_H\mathcal{YD}$, and Hopf algebras B with morphisms of Hopf algebras $f : H \rightarrow B$, $g : B \rightarrow H$ such that $g \circ f = id_H$. Later on, Bespalov proved the same result for braided categories with split idempotents in [5], and pursued further the development of Radford’s theory in joint work with Drabant.

In [1] Alonso and Fernández found a very short proof of Radford’s result using the notion of H -cleft comodule (module) algebras (coalgebras) for a Hopf algebra H in a braided monoidal category. In some sense the approach of [1] is motivated by a characterization of crossed products as a sort of cleft extensions and normal Galois extensions and it can be extended to the theory of entwined modules. In [20], using these ideas, Fernández and Rodríguez obtained, for a C -(co)cleft extension A in the braided category \mathcal{C} , a cross (co)product semi(co)algebra in \mathcal{C} . As a direct consequence, they give an example of a crossed product by a coalgebra coming from the theory of coalgebra bundles and an application to the case of two morphisms $f : C \rightarrow A$ and $g : A \rightarrow C$ of algebras and coalgebras satisfying certain conditions. In particular, if A and C are Hopf algebras, then we have a braided interpretation of Radford’s theorem, i.e., the results of Majid and Bespalov.

The main goal of this paper is to find a good definition of cleft extension for weak entwining structures and with it to obtain a general theory involving as a particular instances the results of the last two paragraphs. The organization of our paper is the following. In Section 1, for a weak entwining structure, we introduce the notion of weak C -cleft extension $A_C \hookrightarrow A$, being A an algebra, C a coalgebra and A_C a subalgebra of A . We prove that if A is a weak C -cleft extension, then there is an isomorphism of algebras between A and a cross product algebra of A_C and C where the base object of this crossed product is a subobject of the tensor product between A_C and C in the braided category \mathcal{C} . In the entwining case we recover the results of [20] and we also give an application of this theory to weak Hopf algebras with coalgebra splitting obtaining, for example, the theory developed in [3]. In Section 2, we prove the dual results and linking the information of this two sections, in Section 3, we obtain an application to the case of two morphisms $f : C \rightarrow A$ and $g : A \rightarrow C$ of algebras and coalgebras satisfying certain conditions. As examples, we obtain the results of [20] and in the case of Hopf algebras with projection we have the results of Majid, Bespalov (braided categories). Finally, if we work with weak Hopf algebras in symmetric categories with split idempotents we obtain the results of [3]. Of course, we have as a particular example Radford's theorem.

1. Weak entwining structures and weak C -cleft extensions

We assume that the reader is familiar with the machinery of braided monoidal categories. Details may be found in [21]. In what follows we denote with $(\mathcal{C}, \otimes, K, c)$ a strict braided monoidal category with equalizers and coequalizers. It is an easy exercise to prove that if we have equalizers and coequalizers, then there exist split idempotents, i.e., for every morphism $q : Y \rightarrow Y$ such that $q = q \circ q$, there exist an object Z and morphisms $i : Z \rightarrow Y$ and $p : Y \rightarrow Z$ verifying $q = i \circ p$ and $p \circ i = id_Z$.

An algebra in \mathcal{C} is a triple $A = (A, \eta_A, \mu_A)$ where A is an object in \mathcal{C} and $\eta_A : K \rightarrow A$ (unit), $\mu_A : A \otimes A \rightarrow A$ (product) are morphisms in \mathcal{C} such that $\mu_A \circ (A \otimes \eta_A) = id_A = \mu_A \circ (\eta_A \otimes A)$, $\mu_A \circ (A \otimes \mu_A) = \mu_A \circ (\mu_A \otimes A)$. Given two algebras $A = (A, \eta_A, \mu_A)$ and $B = (B, \eta_B, \mu_B)$, $f : A \rightarrow B$ is an algebra morphism if $\mu_B \circ (f \otimes f) = f \circ \mu_A$, $f \circ \eta_A = \eta_B$. Also, if A, B are algebras in \mathcal{C} , the object $A \otimes B$ is also an algebra in \mathcal{C} where $\eta_{A \otimes B} = \eta_A \otimes \eta_B$ and $\mu_{A \otimes B} = (\mu_A \otimes \mu_B) \circ (A \otimes c_{B,A} \otimes B)$.

A coalgebra in \mathcal{C} is a triple $D = (D, \varepsilon_D, \delta_D)$ where D is an object in \mathcal{C} and $\varepsilon_D : D \rightarrow K$ (counit), $\delta_D : D \rightarrow D \otimes D$ (coproduct) are morphisms in \mathcal{C} such that $(\varepsilon_D \otimes D) \circ \delta_D = id_D = (D \otimes \varepsilon_D) \circ \delta_D$, $(\delta_D \otimes D) \circ \delta_D = (D \otimes \delta_D) \circ \delta_D$. If $D = (D, \varepsilon_D, \delta_D)$ and $E = (E, \varepsilon_E, \delta_E)$ are coalgebras, $f : D \rightarrow E$ is a coalgebra morphism if $(f \otimes f) \circ \delta_D = \delta_E \circ f$, $\varepsilon_E \circ f = \varepsilon_D$. When D, E are coalgebras in \mathcal{C} , $D \otimes E$ is a coalgebra in \mathcal{C} where $\varepsilon_{D \otimes E} = \varepsilon_D \otimes \varepsilon_E$ and $\delta_{D \otimes E} = (D \otimes c_{D,E} \otimes E) \circ (\delta_D \otimes \delta_E)$.

Definition 1.1. A right–right weak entwining structure on \mathcal{C} consists of a triple (A, C, ψ) , where A is an algebra, C a coalgebra, and $\psi : C \otimes A \rightarrow A \otimes C$ a morphism satisfying the relations

$$(i) \quad \psi \circ (C \otimes \mu_A) = (\mu_A \otimes C) \circ (A \otimes \psi) \circ (\psi \otimes A),$$

- (ii) $(A \otimes \delta_C) \circ \psi = (\psi \otimes C) \circ (C \otimes \psi) \circ (\delta_C \otimes A),$
- (iii) $\psi \circ (C \otimes \eta_A) = (e_{RR} \otimes C) \circ \delta_C,$
- (iv) $(A \otimes \varepsilon_C) \circ \psi = \mu_A \circ (e_{RR} \otimes A),$

where $e_{RR}: C \rightarrow A$ is the morphism defined by $e_{RR} = (A \otimes \varepsilon_C) \circ \psi \circ (C \otimes \eta_A)$. The morphism ψ is called interwinning.

The definition of right–right weak entwining structure was introduced by Caenepeel and de Groot in [16] and is a generalization of the notion of right–right entwining structure defined by Brzeziński and Majid in [14,15]. In the definition of these authors the morphism $e_{RR} = \eta_A \otimes \varepsilon_C$ and, obviously, any right–right entwining structure is a right–right weak entwining structure. Moreover, a right–right weak entwining structure is a right–right entwining structure if and only if $e_{RR} = \eta_A \otimes \varepsilon_C$. Also, in a similar way, we can define the notions of right–left, left–right and left–left weak entwining structures (see [16] for the details).

In this paper we only work with right–right weak entwining structures. For more simplicity we use the name weak entwining structure for substitution of right–right weak entwining structure.

Examples 1.2. (i) Let \mathcal{C} be a symmetric monoidal category with split idempotents. Weak Hopf algebras are generalizations of Hopf algebras and was introduced by Böhm, Nill and Szlachányi in [11,12]. The axioms are the same as the ones for a Hopf algebra, except that the coproduct of the unit, the product of the counit and the antipode condition are replaced by weaker properties. The definition is the following.

A weak Hopf algebra H in \mathcal{C} is an algebra (H, η_H, μ_H) and coalgebra $(H, \varepsilon_H, \delta_H)$ such that the following axioms hold:

- (a1) $\delta_H \circ \mu_H = (\mu_H \otimes \mu_H) \circ \delta_{H \otimes H}.$
- (a2) $\varepsilon_H \circ \mu_H \circ (\mu_H \otimes H) = (\varepsilon_H \otimes \varepsilon_H) \circ (\mu_H \otimes \mu_H) \circ (H \otimes \delta_H \otimes H) = (\varepsilon_H \otimes \varepsilon_H) \circ (\mu_H \otimes \mu_H) \circ (H \otimes (c_{H,H} \circ \delta_H) \otimes H).$
- (a3) $(\delta_H \otimes H) \circ \delta_H \circ \eta_H = (H \otimes \mu_H \otimes H) \circ (\delta_H \otimes \delta_H) \circ (\eta_H \otimes \eta_H) = (H \otimes (\mu_H \circ c_{H,H}) \otimes H) \circ (\delta_H \otimes \delta_H) \circ (\eta_H \otimes \eta_H).$
- (a4) There exists a morphism $\lambda_H: H \rightarrow H$ in \mathcal{C} (called antipode of H) verifying:
 - (a4-1) $\mu_H \circ (H \otimes \lambda_H) \circ \delta_H = ((\varepsilon_H \circ \mu_H) \otimes H) \circ (H \otimes c_{H,H}) \circ ((\delta_H \circ \eta_H) \otimes H).$
 - (a4-2) $\mu_H \circ (\lambda_H \otimes H) \circ \delta_H = (H \otimes (\varepsilon_H \circ \mu_H)) \circ (c_{H,H} \otimes H) \circ (H \otimes (\delta_H \circ \eta_H)).$
 - (a4-3) $\mu_H \circ (\mu_H \otimes H) \circ (\lambda_H \otimes H \otimes \lambda_H) \circ (\delta_H \otimes H) \circ \delta_H = \lambda_H.$

As a consequence of this definition it is an easy exercise to prove that a weak Hopf algebra is a Hopf algebra if and only if the morphism δ_H (coproduct) is unit-preserving (i.e., $\eta_H \otimes \eta_H = \delta_H \circ \eta_H$) and if and only if the counit is a homomorphism of algebras (i.e., $\varepsilon_H \circ \mu_H = \varepsilon_H \otimes \varepsilon_H$).

If H is a weak Hopf algebra, the antipode λ_H is unique, antimultiplicative, anticomultiplicative and leaves the unit η_H and the counit ε_H invariant, i.e., $\lambda_H \circ \mu_H = \mu_H \circ (\lambda_H \otimes \lambda_H) \circ c_{H,H}$, $\delta_H \circ \lambda_H = c_{H,H} \circ (\lambda_H \otimes \lambda_H) \circ \delta_H$, $\lambda_H \circ \eta_H = \eta_H$, $\varepsilon_H \circ \lambda_H = \varepsilon_H$.

If we define the morphisms Π_H^L , Π_H^R , $\overline{\Pi}_H^L$ and $\overline{\Pi}_H^R$ by

$$\begin{aligned}
\Pi_H^L &= ((\varepsilon_H \circ \mu_H) \otimes H) \circ (H \otimes c_{H,H}) \circ ((\delta_H \circ \eta_H) \otimes H) : H \rightarrow H, \\
\Pi_H^R &= (H \otimes (\varepsilon_H \circ \mu_H)) \circ (c_{H,H} \otimes H) \circ (H \otimes (\delta_H \circ \eta_H)) : H \rightarrow H, \\
\overline{\Pi}_H^L &= (H \otimes (\varepsilon_H \circ \mu_H)) \circ ((\delta_H \circ \eta_H) \otimes H) : H \rightarrow H, \\
\overline{\Pi}_H^R &= ((\varepsilon_H \circ \mu_H) \otimes H) \circ (H \otimes (\delta_H \circ \eta_H)) : H \rightarrow H.
\end{aligned}$$

It is straightforward to show (see [11]) that they are idempotent. Moreover, we have that (see [16]) $\overline{\Pi}_H^R \circ \Pi_H^L = \Pi_H^L$, $\Pi_H^L \circ \overline{\Pi}_H^R = \overline{\Pi}_H^R$, $\overline{\Pi}_H^L \circ \Pi_H^R = \Pi_H^R$, $\Pi_H^R \circ \overline{\Pi}_H^L = \overline{\Pi}_H^L$, $\overline{\Pi}_H^L \circ \Pi_H^L = \overline{\Pi}_H^L$, $\Pi_H^L \circ \overline{\Pi}_H^L = \Pi_H^L$, $\overline{\Pi}_H^R \circ \Pi_H^R = \overline{\Pi}_H^R$ and $\Pi_H^R \circ \overline{\Pi}_H^R = \Pi_H^R$. Also it is easy to show the formulas:

$$\begin{aligned}
\Pi_H^L &= \overline{\Pi}_H^R \circ \lambda_H = \lambda_H \circ \overline{\Pi}_H^L, & \Pi_H^R &= \overline{\Pi}_H^L \circ \lambda_H = \lambda_H \circ \overline{\Pi}_H^R, \\
\Pi_H^L \circ \lambda_H &= \Pi_H^L \circ \Pi_H^R = \lambda_H \circ \Pi_H^R, & \Pi_H^R \circ \lambda_H &= \Pi_H^R \circ \Pi_H^L = \lambda_H \circ \Pi_H^L.
\end{aligned}$$

A morphism between weak Hopf algebras H and B is a morphism $f : H \rightarrow B$ which is both algebra and coalgebra morphism. If $f : H \rightarrow B$ is a weak Hopf algebra morphism, then $\lambda_B \circ f = f \circ \lambda_H$ (see 1.4 of [2]).

Let be the triple (H, H, ψ) where $\psi = (H \otimes \mu_H) \circ (c_{H,H} \otimes H) \circ (H \otimes \delta_H)$. Then (H, H, ψ) is a weak entwining structure with $e_{RR} = \Pi_H^R$.

(ii) Let H, B be weak Hopf algebras in a symmetric monoidal category \mathcal{C} with split idempotents. Let $g : B \rightarrow H$ be a morphism of weak Hopf algebras and $f : H \rightarrow B$ be a morphism of coalgebras such that $g \circ f = id_H$ and $f \circ \eta_H = \eta_B$. If we define $\rho_B : B \rightarrow B \otimes H$ and the intertwining $\psi : H \otimes B \rightarrow B \otimes H$ by

$$\rho_B = (B \otimes g) \circ \delta_B, \quad \psi = (B \otimes \mu_H) \circ (c_{H,B} \otimes H) \circ (H \otimes \rho_B)$$

we have that (B, H, ψ) is a weak entwining structure where $e_{RR} = \Pi_B^R \circ f$. Of course, the previous example is a particular instance of this one for $H = B$ and $f = g = id_H$.

In [3], using the idempotent morphism $q_H^B = \mu_B \circ (B \otimes (\lambda_B \circ f \circ g)) \circ \delta_B : B \rightarrow B$ and an equalizer diagram, Alonso and González proved that it is possible to construct an algebra B_H and morphisms $\varphi_{B_H} : H \otimes B_H \rightarrow B_H$, $\sigma_{B_H} : H \otimes H \rightarrow B_H$ such that there exists a subobject $B_H \times H$ of $B_H \otimes H$ isomorphic with B as algebras and with algebra structure (the crossed product) defined by $\eta_{B_H \times H} = r_B \circ (\eta_{B_H} \otimes \eta_H)$ and

$$\begin{aligned}
\mu_{B_H \times H} &= r_B \circ (\mu_{B_H} \otimes H) \circ (\mu_{B_H} \otimes \sigma_{B_H} \otimes \mu_H) \circ (B_H \otimes \varphi_{B_H} \otimes \delta_{H \otimes H}) \\
&\quad \circ (B_H \otimes H \otimes c_{H,B_H} \otimes H) \circ (B_H \otimes \delta_H \otimes B_H \otimes H) \circ (s_B \otimes s_B),
\end{aligned}$$

where s_B is the inclusion of $B_H \times H$ in $B_H \otimes H$ and r_B the projection of $B_H \otimes H$ on $B_H \times H$. Of course, when $f : H \rightarrow B$ is a morphism of weak Hopf algebras we recover the theory developed in [2] and if H and B are Hopf algebras we obtain the result of Blattner, Cohen and Montgomery (see [9]). For this reason, the authors denoted the algebra $B_H \times H$ by $B_H \sharp_{\sigma_{B_H}} H$.

In this paper, we will prove that these results are particular instances of a more general theorems that we can obtain in the context of weak entwining structures.

Definition 1.3. Let (A, C, ψ) be a weak entwining structure in \mathcal{C} . We denote by $\mathcal{M}_A^C(\psi)$ the category whose objects are triples (M, ϕ_M, ρ_M) , where (M, ϕ_M) is a right A -module (i.e., $\phi_M \circ (\phi_M \otimes A) = \phi_M \circ (M \otimes \mu_A)$, $id_M = \phi_M \circ (M \otimes \eta_A)$), (M, ρ_M) is a right C -comodule (i.e., $(\rho_M \otimes C) \circ \rho_M = (M \otimes \delta_C) \circ \rho_M$, $(M \otimes \varepsilon_C) \circ \rho_M = id_M$), and

$$\rho_M \circ \phi_M = (\phi_M \otimes C) \circ (M \otimes \psi) \circ (\rho_M \otimes A).$$

The objects of $\mathcal{M}_A^C(\psi)$ will be called weak entwined modules and a morphism in $\mathcal{M}_A^C(\psi)$ is a morphism of A -modules and C -comodules. If (A, C, ψ) is an entwining structure then we find the category of entwined modules introduced by Brzeziński in [14]. Finally, in a similar way we define modules over right–left, left–right and left–left weak entwining structures (see [16]).

Examples 1.4. (i) Using entwining structures it is possible to unify some categories of modules associated to a Hopf algebra as categories of entwined modules. For example, if $C = H$ is a Hopf algebra, A is a right H -comodule algebra and $\psi = (A \otimes \mu_H) \circ (c_{H,A} \otimes H) \circ (H \otimes \rho_A)$ an object M in $\mathcal{M}_A^H(\psi)$ is a Hopf module [17]. If $C = A = H$ is a Hopf algebra and

$$\begin{aligned} \psi &= (H \otimes \mu_H) \circ (H \otimes H \otimes \mu_H) \circ (H \otimes c_{H,H} \otimes H) \circ (c_{H,H} \otimes H \otimes H) \\ &\circ (H \otimes c_{H,H} \otimes H) \circ (H \otimes \lambda_H \otimes \delta_H) \circ (H \otimes \delta_H) \end{aligned}$$

then an object M in $\mathcal{M}_A^C(\psi)$ is a Yetter–Drinfeld module [28,29]. Finally, let H be a Hopf algebra. If A is a right H -comodule algebra, C is a right H -module coalgebra and $\psi = (A \otimes \phi_C) \circ (c_{C,A} \otimes H) \circ (C \otimes \rho_A)$ then an object in $\mathcal{M}_A^C(\psi)$ is a Doi–Koppinen module [18,22].

(ii) If H and B are weak Hopf algebras in the same conditions of (ii) of 1.2 then $(B, \phi_B = \mu_B, \rho_B)$ belongs to the category $\mathcal{M}_B^H(\psi)$.

(iii) The category of weak Doi–Hopf modules, introduced in [10] can be identify as a category of weak entwined modules (see [16]).

Proposition 1.5. Let (A, C, ψ) be a weak entwining structure such that there exists a coaction ρ_A verifying that (A, μ_A, ρ_A) belongs to $\mathcal{M}_A^C(\psi)$. If for all $(M, \phi_M, \rho_M) \in \mathcal{M}_A^C(\psi)$, we denote by M_C the equalizer of ρ_M and $\zeta_M = (\phi_M \otimes C) \circ (M \otimes (\rho_A \circ \eta_A))$ and by i_C^M the injection of M_C in M , we have the following:

- (i) The triple $(A_C, \eta_{A_C}, \mu_{A_C})$ is an algebra in \mathcal{C} , where $\eta_{A_C} : K \rightarrow A_C$ and $\mu_{A_C} : A_C \otimes A_C \rightarrow A_C$ are the factorizations of η_A and $\mu_A \circ (i_C^A \otimes i_C^A)$ respectively, through the equalizer i_C^A .
- (ii) The pair (M_C, ϕ_{M_C}) is a right A_C -module, where $\phi_{M_C} : M_C \otimes A_C \rightarrow M_C$ is the factorization of $\phi_M \circ (i_C^M \otimes i_C^A)$ through the equalizer i_C^M .

Proof. The proof is easy and we leave the details to the reader. \square

Example 1.6. If H and B are weak Hopf algebras in the same conditions of (ii) of 1.2 then the morphism $q_H^B = \mu_B \circ (B \otimes (\lambda_B \circ f \circ g)) \circ \delta_B : B \rightarrow B$ is an idempotent in \mathcal{C} (see 2.1 of [3]). As a consequence, there exist an epimorphism p_H^B , a monomorphism i_H^B and an object B_H such that the diagram

$$\begin{array}{ccc} B & \xrightarrow{q_H^B} & B \\ & \searrow p_H^B & \nearrow i_H^B \\ & B_H & \end{array}$$

commutes and $p_H^B \circ i_H^B = id_{B_H}$. Also by 2.2 of [3] we have that the following diagram is an equalizer diagram in \mathcal{C} :

$$B_H \xrightarrow{i_H^B} B \begin{array}{c} \xrightarrow{\rho_B} \\ \xrightarrow{(B \otimes \Pi_H^L) \circ \rho_B} \end{array} B \otimes H.$$

Moreover, using the equality $\overline{\Pi}_H^R \circ \Pi_H^L = \Pi_H^L$, it is easy to show that

$$B_H \xrightarrow{i_H^B} B \begin{array}{c} \xrightarrow{\rho_B} \\ \xrightarrow{(B \otimes \overline{\Pi}_H^R) \circ \rho_B} \end{array} B \otimes H$$

is an equalizer diagram in \mathcal{C} .

Therefore, the object defined by the equalizer of ρ_B and ζ_B is the same that the one defined by the equalizer of ρ_B and $(B \otimes \overline{\Pi}_H^R) \circ \rho_B$ because, in this situation, $(B \otimes \overline{\Pi}_H^R) \circ \rho_B = \zeta_B$.

Remark 1.7. Suppose that (A, C, ψ) be a weak entwining structure such that there exists a coaction ρ_A verifying that (A, μ_A, ρ_A) belongs to $\mathcal{M}_A^C(\psi)$. Then if $h \in Hom_{\mathcal{C}}(C, A)$ is a morphism of right C -comodules $h \wedge e_{RR} = h$ where \wedge denotes the usual convolution, i.e., $h \wedge e_{RR} = \mu_A \circ (h \otimes e_{RR}) \circ \delta_C$. Indeed:

$$\begin{aligned} h \wedge e_{RR} &= (\mu_A \otimes \varepsilon_C) \circ (h \otimes \psi) \circ (\delta_C \otimes \eta_A) \\ &= (\mu_A \otimes \varepsilon_C) \circ (A \otimes \psi) \circ ((\rho_A \circ h) \otimes \eta_A) \\ &= (A \otimes \varepsilon_C) \circ \rho_A \circ \mu_A \circ (h \otimes \eta_A) = h. \end{aligned}$$

Definition 1.8. Let (A, C, ψ) be a weak entwining structure and suppose that (A, ρ_A) is a right C -comodule. By $Reg^{WR}(C, A)$ we denote the set of morphisms $h \in Hom_{\mathcal{C}}(C, A)$ such that there exists a morphism $h^{-1} \in Hom_{\mathcal{C}}(C, A)$ (the left weak inverse of h) verifying $h^{-1} \wedge h = e_{RR}$.

Let A be an algebra and C be a coalgebra in \mathcal{C} . By $Reg(C, A)$ we denote the set of morphisms $h : C \rightarrow A$ such that there exists a morphism $h^{-1} : C \rightarrow A$ (the inverse of h) verifying $h^{-1} \wedge h = h \wedge h^{-1} = \varepsilon_C \otimes \eta_A = \eta_A \circ \varepsilon_C$. Of course, if (A, C, ψ) is an entwining structure in \mathcal{C} $e_{RR} = \varepsilon_C \otimes \eta_A$ and $Reg(C, A) \subset Reg^{WR}(C, A)$.

Definition 1.9. Let (A, C, ψ) be a weak entwining structure and suppose that $(A, \mu_A, \rho_A) \in \mathcal{M}_A^C(\psi)$. We will say that $A_C \hookrightarrow A$ is a weak C -cleft extension if there exists a morphism $h: C \rightarrow A$ in $\text{Reg}^{WR}(C, A)$ of right C -comodules such that

$$\psi \circ (C \otimes h^{-1}) \circ \delta_C = \zeta_A \circ (e_{RR} \wedge h^{-1})$$

where $\zeta_A = (\mu_A \otimes C) \circ (A \otimes (\rho_A \circ \eta_A))$ is the morphism defined in 1.5.

This definition is a generalization of the one used by Fernández and Rodríguez in [20] in the context of entwined modules but changing $\text{Reg}(C, A)$ by $\text{Reg}^{WR}(C, A)$ and adding a new condition. In this paper we will show that the results obtained in [20] can be prove if you only work with the more general set $\text{Reg}^{WR}(C, A)$. Also, using the definition of weak C -cleft extension, we will involve the weak Hopf algebraic context and the weak entwined categories of modules in connection with it.

A classic result in Galois theory says that if $B \subset A$ is a finite Galois extension of fields with Galois group H , then A/B has a normal basis, i.e., there exists $a \in A$ such that the set $\{x.a; x \in H\}$ is a basis for A over B . Afterwards, Kreimer and Takeuchi introduce in [23] the notion of normal basis for extensions, associated to Hopf algebras in categories of modules over a commutative ring, and in [19] Doi and Takeuchi characterized the H -Galois extensions with normal basis in terms of H -cleft extensions. Recently, in the work of Brzeziński [14] we can find a more general formulation of these last results in the context of entwining structures.

In [4], we formulate the definition of weak C -Galois extension with normal basis for a weak entwining structure living in a braided monoidal category with equalizers and coequalizers and we characterize this extensions using the notion of cleftness introduced in Definition 1.9. Of course, as a particular instances, we recover the results described in the previous paragraph.

Remarks 1.10. (i) Suppose that (A, C, ψ) be a weak entwining structure such that there exists a coaction ρ_A verifying that (A, μ_A, ρ_A) belongs to $\mathcal{M}_A^C(\psi)$. Then if $h: C \rightarrow A$ is a morphism of right C -comodules in $\text{Reg}^{WR}(C, A)$, the intertwining ψ is completely determined in the following form:

$$\psi = (\mu_A \otimes C) \circ (A \otimes (\rho_A \circ \mu_A)) \circ (((h^{-1} \otimes h) \circ \delta_C) \otimes A).$$

Indeed:

$$\begin{aligned} & (\mu_A \otimes C) \circ (A \otimes (\rho_A \circ \mu_A)) \circ (((h^{-1} \otimes h) \circ \delta_C) \otimes A) \\ &= (\mu_A \otimes C) \circ (A \otimes ((\mu_A \otimes C) \circ (A \otimes \psi) \circ (\rho_A \otimes A))) \circ (((h^{-1} \otimes h) \circ \delta_C) \otimes A) \\ &= (\mu_A \otimes C) \circ (\mu_A \otimes \psi) \circ (h^{-1} \otimes h \otimes C \otimes A) \circ (C \otimes \delta_C \otimes A) \circ (\delta_C \otimes A) \\ &= (\mu_A \otimes C) \circ (e_{RR} \otimes \psi) \circ (\delta_C \otimes A) \\ &= (A \otimes \varepsilon_C \otimes C) \circ (\psi \otimes C) \circ (C \otimes \psi) \circ (\delta_C \otimes A) \\ &= (A \otimes ((\varepsilon_C \otimes C) \circ \delta_C)) \circ \psi = \psi. \end{aligned}$$

In the last computations, the first equality it is true because A is an entwined module. The second one follows from the fact that h is a morphism of right C -modules, the third one by the coalgebra structure of C and by $h^{-1} \wedge h = e_{RR}$ and finally, the fourth and the fifth ones by the properties of ψ and the coalgebra structure of C .

(ii) Let (A, C, ψ) be an entwined structure and suppose that $(A, \mu_A, \rho_A) \in \mathcal{M}_A^C(\psi)$. If $h \in \text{Reg}(C, A)$ is a morphism of right C -comodules we have that

$$\psi \circ (C \otimes h^{-1}) \circ \delta_C = \zeta_A \circ h^{-1} = \zeta_A \circ (e_{RR} \wedge h^{-1}).$$

Then, as a consequence, a C -cleft extension for an entwining structure is a weak C -cleft extension.

Example 1.11. If H is a Hopf algebra id_H is an element of $\text{Reg}(H, H)$ with inverse $id_H^{-1} = \lambda_H$. In the weak Hopf algebra case $id_H \in \text{Reg}^{WR}(H, H)$ with left weak inverse λ_H and $id_H \in \text{Reg}(H, H)$ if and only if H is a Hopf algebra.

If H and B are weak Hopf algebras in the same conditions of (ii) of 1.2, $f \in \text{Reg}^{WR}(H, B)$ because for $f^{-1} = \lambda_B \circ f$ we obtain that $f^{-1} \wedge f = \Pi_B^R \circ f = e_{RR}$ and $B_H \hookrightarrow B = (B, \mu_B, \rho_B)$ is a weak H -cleft extension because

$$\begin{aligned} & \psi \circ (H \otimes f^{-1}) \circ \delta_H \\ &= (B \otimes \mu_H) \circ (c_{H,B} \otimes H) \circ (H \otimes ((\lambda_B \otimes (g \circ \lambda_B)) \circ c_{B,B} \circ \delta_B \circ f)) \circ \delta_H \\ &= (B \otimes g) \circ c_{B,B} \circ (\Pi_B^L \otimes \lambda_B) \circ \delta_B \circ f \\ &= (B \otimes (g \circ \overline{\Pi}_B^R)) \circ c_{B,B} \circ (\Pi_B^L \otimes \lambda_B) \circ \delta_B \circ f \\ &= (B \otimes \overline{\Pi}_H^R) \circ \rho_B \circ f^{-1} \\ &= \zeta_B \circ f^{-1} \\ &= \zeta_B \circ (\Pi_B^R \wedge \lambda_B) \circ f \\ &= \zeta_B \circ (e_{RR} \wedge f^{-1}). \end{aligned}$$

In the previous calculus, the first equality follows from the definition of ψ and by the antimultiplicative nature of λ_B . In the second one, we use the equality $g \circ f = i_H$, the condition of coalgebra morphism for f and the naturality of the braiding. The third equality follows from $\Pi_B^L = \overline{\Pi}_B^R \circ \lambda_B$ and in the fourth one we use the antimultiplicative nature of the antipode. In the fifth one we apply $\overline{\Pi}_H^R \circ g = g \circ \overline{\Pi}_B^R$ and $(B \otimes \overline{\Pi}_H^R) \circ \rho_B = \zeta_B$. Finally, in the sixth one we use $\Pi_B^R \wedge \lambda_B = \lambda_B$ and in the seventh one $(\Pi_B^R \wedge \lambda_B) \circ f = e_{RR} \wedge f^{-1}$.

When $H = B$ and $f = g = id_H$ we have the trivial example of weak H -cleft extension.

Proposition 1.12. Let (A, C, ψ) be a weak entwining structure such that there exists a coaction ρ_A verifying that (A, μ_A, ρ_A) belongs to $\mathcal{M}_A^C(\psi)$. Let $h : C \rightarrow A$ be a morphism of right C -comodules in $\text{Reg}^{WR}(C, A)$. The following are equivalent:

- (i) $A_C \hookrightarrow A$ is a weak C -cleft extension.

- (ii) For all object M in $\mathcal{M}_A^C(\psi)$ the morphism $q_C^M = \phi_M \circ (M \otimes h^{-1}) \circ \rho_M : M \rightarrow M$ factors through the equalizer i_C^M , i.e., there exists a morphism $p_C^M : M \rightarrow M_C$ such that $i_C^M \circ p_C^M = q_C^M$.

Proof. (i) \Rightarrow (ii). We have that

$$\begin{aligned}
 \rho_M \circ q_C^M &= (\phi_M \otimes C) \circ (M \otimes \psi) \circ (\rho_M \otimes h^{-1}) \circ \rho_M \\
 &= (\phi_M \otimes C) \circ (M \otimes (\psi \circ (C \otimes h^{-1}) \circ \delta_C)) \circ \rho_M \\
 &= (\phi_M \otimes C) \circ (M \otimes (\zeta_A \circ (e_{RR} \wedge h^{-1}))) \circ \rho_M \\
 &= (\phi_M \otimes C) \circ ((\phi_M \circ (M \otimes e_{RR}) \circ \rho_M) \otimes (\zeta_A \circ h^{-1})) \circ \rho_M \\
 &= (\phi_M \otimes C) \circ ((\phi_M \circ (M \otimes h^{-1}) \circ \rho_M) \otimes (\rho_A \circ \eta_A)) \\
 &= \zeta_M \circ q_C^M.
 \end{aligned}$$

In the last computations the first equality follows from the weak entwined condition for M , the second one by the comodule structure of M and the third one by the cleft condition. In the fourth one we use the comodule structure of M and in the fifth one we apply $\phi_M \circ (M \otimes e_{RR}) \circ \rho_M = id_M$. Finally, in the sixth one we use the module structure of M .

Therefore, there exists a morphism $p_C^M : M \rightarrow M_C$ such that $i_C^M \circ p_C^M = q_C^M$.

(ii) \Rightarrow (i). For $M = A$ we have the following:

$$\begin{aligned}
 \psi \circ (C \otimes h^{-1}) \circ \delta_C &= (\mu_A \otimes C) \circ (A \otimes (\rho_A \circ \mu_A)) \circ (((h^{-1} \otimes h) \circ \delta_C) \otimes A) \circ (C \otimes h^{-1}) \circ \delta_C \\
 &= (\mu_A \otimes C) \circ (A \otimes (\rho_A \circ q_C^A)) \circ (h \otimes h^{-1}) \circ \delta_C \\
 &= (\mu_A \otimes C) \circ (h^{-1} \otimes ((\mu_A \otimes C) \circ (\mu_A \otimes (\rho_A \circ \eta_A)) \circ (A \otimes h^{-1}) \circ \rho_A \circ h)) \circ \delta_C \\
 &= (\mu_A \otimes C) \circ (\mu_A \otimes (\rho_A \circ \eta_A)) \circ (\mu_A \otimes A) \circ (h^{-1} \otimes h \otimes h^{-1}) \circ (\delta_C \otimes C) \circ \delta_C \\
 &= \zeta_A \circ (e_{RR} \wedge h^{-1}).
 \end{aligned}$$

Note that in the last computations, the first follows by the expression of ψ calculated in 1.10 and the third one by the factorization condition for q_C^A . \square

Remark 1.13. Let (A, C, ψ) be a weak entwining structure such that there exists a coaction ρ_A verifying that (A, μ_A, ρ_A) belongs to $\mathcal{M}_A^C(\psi)$. If $g : C \rightarrow A$ is a morphism in \mathcal{C} , the following assertions are equivalent:

- (a1) The morphism g verifies $e_{RR} \wedge g = g$ and $\psi \circ (C \otimes g) \circ \delta_C = \zeta_A \circ (e_{RR} \wedge g)$.
- (a2) The morphism g verifies $\psi \circ (C \otimes g) \circ \delta_C = \zeta_A \circ g$.

Indeed, (a1) \Rightarrow (a2) is trivial. On the other hand, (a2) \Rightarrow (a1) is true because:

$$g = (\mu_A \otimes \varepsilon_C) \circ (g \otimes \rho_A) \circ (C \otimes \eta_A) = (A \otimes \varepsilon_C) \circ \psi \circ (C \otimes g) \circ \delta_C = e_{RR} \wedge g.$$

Let $A_C \hookrightarrow A$ be a weak C -cleft extension with associated morphism $h : C \rightarrow A$. The morphism $g = e_{RR} \wedge h^{-1}$ verifies the following:

- (b1) $g \wedge h = e_{RR}$.
- (b2) $e_{RR} \wedge g = g$.
- (b3) $\psi \circ (C \otimes g) \circ \delta_C = \zeta_A \circ (e_{RR} \wedge g)$.

Indeed, (b1), (b2) are trivial. The proof for (b3) is the following:

$$\begin{aligned} \psi \circ (C \otimes g) \circ \delta_C &= (\mu_A \otimes C) \circ (h^{-1} \otimes \rho_A) \circ (C \otimes h \wedge e_{RR} \wedge h^{-1}) \circ \delta_C \\ &= (\mu_A \otimes C) \circ (h^{-1} \otimes \rho_A) \circ (C \otimes \mu_A) \circ (C \otimes h \otimes h^{-1}) \circ (\delta_C \otimes C) \circ \delta_C \\ &= \psi \circ (C \otimes h^{-1}) \circ \delta_C \\ &= \zeta_A \circ (e_{RR} \wedge h^{-1}) \\ &= \zeta_A \circ (e_{RR} \wedge g). \end{aligned}$$

Then using the equivalence (a1) \Leftrightarrow (a2), we obtain that (b1), (b2) and (b3) are equivalent with (a1) and $\psi \circ (C \otimes g) \circ \delta_C = \zeta_A \circ g$. Therefore, in Definition 1.9 we can suppose without loss of generality that $e_{RR} \wedge h^{-1} = h^{-1}$.

Example 1.14. If H and B are weak Hopf algebras in the same conditions of (ii) of 1.2 then the morphism introduced in the last proposition is the morphism q_H^B defined in 1.6. The morphism q_H^B was used by Bespalov [5], in the context of braided categories with split idempotents, for to obtain a braided version of the Radford's theorem (see [27]) for Hopf algebras with projection (see also [6–8]). The first result in this direction was stated by Majid [24] using the notion of Yetter–Drinfeld modules and the bosonization process.

Proposition 1.15. Let $A_C \hookrightarrow A$ be a weak C -cleft extension with $h : C \rightarrow A$ the morphism of right C -comodules in $\text{Reg}^{WR}(C, A)$. Then the morphism $\varphi_A : C \otimes A \rightarrow A$ defined by

$$\varphi_A = \mu_A \circ (\mu_A \otimes h^{-1}) \circ (h \otimes \psi) \circ (\delta_C \otimes A)$$

factors through the equalizer i_C^A . Moreover, if φ'_A is the factorization of φ_A , we have the following equality:

$$\mu_{AC} \circ (\varphi'_A \otimes \varphi'_A) \circ (C \otimes \psi \otimes A) \circ (\delta_C \otimes A \otimes A) = \varphi'_A \circ (C \otimes \mu_A).$$

Finally, if we define the morphism $\varphi_{A_C} : C \otimes A_C \rightarrow A_C$ by $\varphi_{A_C} = \varphi'_A \circ (C \otimes i_A^C)$ we obtain:

$$\mu_{A_C} \circ (\varphi'_A \otimes \varphi_{A_C}) \circ (C \otimes \psi \otimes A_C) \circ (\delta_C \otimes i_A^C \otimes A_C) = \varphi_{A_C} \circ (C \otimes \mu_{A_C}).$$

Proof. Let $\varphi_A : C \otimes A \rightarrow A$ be the morphism defined by $\varphi_A = \mu_A \circ (\mu_A \otimes h^{-1}) \circ (h \otimes \psi) \circ (\delta_C \otimes A)$. Then if we put $\varphi'_A = p_C^A \circ \mu_A \circ (h \otimes A)$, where p_C^A is the unique morphism such that $i_C^A \circ p_C^A = q_C^A$, we obtain that $i_C^A \circ \varphi'_A = \varphi_A$.

On the other hand,

$$\begin{aligned} & i_C^A \circ \mu_{A_C} \circ (\varphi'_A \otimes \varphi'_A) \circ (C \otimes \psi \otimes A) \circ (\delta_C \otimes A \otimes A) \\ &= \mu_A \circ (\mu_A \otimes \mu_A) \circ (\mu_A \otimes h^{-1} \otimes \mu_A \otimes h^{-1}) \circ (h \otimes \psi \otimes h \otimes \psi) \\ & \quad \circ (\delta_C \otimes A \otimes \delta_C \otimes A) \circ (C \otimes \psi \otimes A) \circ (\delta_C \otimes A \otimes A) \\ &= \mu_A \circ (\mu_A \otimes A) \circ (\mu_A \otimes A \otimes \mu_A) \circ (h \otimes A \otimes e_{RR} \otimes A \otimes h^{-1}) \\ & \quad \circ (C \otimes \psi \otimes \psi) \circ (\delta_C \otimes \psi \otimes A) \circ (\delta_C \otimes A \otimes A) \\ &= \mu_A \circ (\mu_A \otimes h^{-1}) \circ (\mu_A \otimes \psi) \circ (\mu_A \otimes e_{RR} \otimes C \otimes A) \circ (h \otimes A \otimes \delta_C \otimes A) \\ & \quad \circ (C \otimes \psi \otimes A) \circ (\delta_C \otimes A \otimes A) \\ &= \mu_A \circ (\mu_A \otimes h^{-1}) \circ (\mu_A \otimes \mu_A \otimes C) \circ (h \otimes A \otimes A \otimes \psi) \circ (C \otimes A \otimes \psi \otimes A) \\ & \quad \circ (C \otimes \psi \otimes A \otimes A) \circ (\delta_C \otimes A \otimes \eta_A \otimes A) \\ &= \mu_A \circ (\mu_A \otimes h^{-1}) \circ (h \otimes \psi) \circ (\delta_C \otimes \mu_A) \\ &= i_C^A \circ \varphi'_A \circ (C \otimes \mu_A). \end{aligned}$$

In the last computations, the first equality follows from $i_C^A \circ \varphi'_A = \varphi_A$ and by the definition of φ_A , the second one by the properties of ψ and the associativity of A , the third, the fourth and the fifth ones by the properties of ψ . Finally, the sixth equality is trivial.

Therefore,

$$\mu_{A_C} \circ (\varphi'_A \otimes \varphi'_A) \circ (C \otimes \psi \otimes A) \circ (\delta_C \otimes A \otimes A) = \varphi'_A \circ (C \otimes \mu_A).$$

Finally, if we compose in this equality with $C \otimes i_C^A \otimes i_C^A$, we have that

$$\mu_{A_C} \circ (\varphi'_A \otimes \varphi_{A_C}) \circ (C \otimes \psi \otimes A_C) \circ (\delta_C \otimes i_A^C \otimes A_C) = \varphi_{A_C} \circ (C \otimes \mu_{A_C}). \quad \square$$

Examples 1.16. (i) In Proposition 2.10 of [20] we can find a similar result in the context of C -cleft extensions for entwining structures.

(ii) Let H, B be weak Hopf algebras in a symmetric monoidal category \mathcal{C} with split idempotents. Let $g : B \rightarrow H$ be a morphism of weak Hopf algebras and $f : H \rightarrow B$ be a morphism of coalgebras such that $g \circ f = id_H$ and $f \circ \eta_H = \eta_B$. In these conditions φ_{B_H} is the morphism defined in Proposition 2.4 of [3]. The morphism φ_{B_H} verifies

- (1) $\varphi_{B_H} \circ (\eta_H \otimes B_H) = id_{B_H}$.
- (2) $\varphi_{B_H} \circ (H \otimes \eta_{B_H}) = \varphi_{B_H} \circ (\Pi_H^L \otimes \eta_{B_H})$.
- (3) $\mu_{B_H} \circ (\varphi_{B_H} \otimes B_H) \circ (H \otimes \eta_{B_H} \otimes B_H) = \varphi_{B_H} \circ (\Pi_H^L \otimes B_H)$.
- (4) $\varphi_{B_H} \circ (H \otimes \mu_{B_H}) = \mu_{B_H} \circ (\varphi_{B_H} \otimes \varphi_{B_H}) \circ (H \otimes c_{H, B_H} \otimes B_H) \circ (\delta_H \otimes B_H \otimes B_H)$.
- (5) $\mu_{B_H} \circ c_{B_H, B_H} \circ ((\varphi_{B_H} \circ (H \otimes \eta_{B_H})) \otimes B_H) = \varphi_{B_H} \circ (\overline{\Pi}_H^L \otimes B_H)$.

By 1.15 we can add to the last equalities the new property

$$(6) \quad \varphi_{B_H} \circ (H \otimes \mu_{B_H}) = \mu_{B_H} \circ (\varphi'_B \otimes \varphi_{B_H}) \circ (H \otimes \psi \otimes B_H) \circ (\delta_H \otimes i_H^B \otimes B_H),$$

where ψ is the entwining defined in (ii) of 1.2.

Moreover, if f is a morphism of algebras (B_H, φ_{B_H}) is a left H -module (see Proposition 2.5 of [2]).

Proposition 1.17. *Let $A_C \hookrightarrow A$ be a weak C -cleft extension with $h : C \rightarrow A$ the morphism of right C -comodules in $\text{Reg}^{WR}(C, A)$. Then the morphism $\sigma_A : C \otimes C \rightarrow A$ defined by*

$$\sigma_A = \mu_A \circ (\mu_A \otimes h^{-1}) \circ (h \otimes \psi) \circ (\delta_C \otimes h)$$

factors through the equalizer i_C^A . Moreover, if σ_{A_C} is the factorization of σ_A , then

$$\sigma_{A_C} = p_C^A \circ \mu_A \circ (h \otimes h).$$

Proof. We have that $\sigma_A = \varphi_A \circ (C \otimes h)$ where φ_A is the morphism defined in 1.15. Then

$$\rho_A \circ \sigma_A = \rho_A \circ \varphi_A \circ (C \otimes h) = \zeta_A \circ \varphi_A \circ (C \otimes h) = \zeta_A \circ \sigma_A.$$

Therefore, there exists a unique morphism $\sigma_{A_C} : C \otimes C \rightarrow A_C$ such that $i_C^A \circ \sigma_{A_C} = \sigma_A$. The morphism σ_{A_C} verifies that $\sigma_{A_C} = p_C^A \circ \mu_A \circ (h \otimes h)$ because $i_C^A \circ p_C^A \circ \mu_A \circ (h \otimes h) = \sigma_A$. \square

1.18. Let $A_C \hookrightarrow A$ be a weak C -cleft extension with morphism $h \in \text{Reg}^{WR}(C, A)$ and let $M \in \mathcal{M}_A^C(\psi)$. The morphisms

$$\omega_M : M_C \otimes C \rightarrow M, \quad \omega'_M : M \rightarrow M_C \otimes C$$

defined by $\omega_M = \phi_M \circ (i_C^M \otimes h)$ and $\omega'_M = (p_C^M \otimes C) \circ \rho_M$ verify the equality $\omega_M \circ \omega'_M = id_M$ because $\omega_M \circ \omega'_M = \phi_M \circ (M \otimes e_{RR}) \circ \rho_M = id_M$. Notice that the equality $\omega'_M \circ \omega_M = id_{M_C \otimes C}$ it is not true in general. In the following proposition we clarify the meaning of the identity $\omega'_M \circ \omega_M = id_{M_C \otimes C}$.

Proposition 1.19. *Let $A_C \hookrightarrow A$ be a weak C -cleft extension with $h : C \rightarrow A$ the morphism of right C -comodules in $\text{Reg}^{WR}(C, A)$ and let $M \in \mathcal{M}_A^C(\psi)$. Then, $\omega'_M \circ \omega_M = id_{M_C \otimes C}$ if and only if $M_C \otimes \varepsilon_C = p_C^M \circ \omega_M$.*

As a consequence, if $A = M$ we have that $\omega'_A \circ \omega_A = id_{A_C \otimes C}$ if and only if $h \wedge h^{-1} = \varepsilon_C \otimes \eta_A$. Therefore, if $\omega'_A \circ \omega_A = id_{A_C \otimes C}$, the left weak inverse of h is unique.

Proof. First note that:

$$\begin{aligned} \omega'_M \circ \omega_M &= ((p_C^M \circ \phi_M) \otimes C) \circ (M \otimes \psi) \circ ((\rho_M \circ i_C^M) \otimes h) \\ &= ((p_C^M \circ \phi_M) \otimes C) \circ (\phi_M \otimes \psi) \circ (i_C^M \otimes (\rho_A \circ \eta_A) \otimes h) \\ &= ((p_C^M \circ \phi_M) \otimes C) \circ (i_C^M \otimes ((\mu_A \otimes C) \circ (A \otimes \psi) \circ ((\rho_A \circ \eta_A) \otimes h))) \\ &= ((p_C^M \circ \phi_M) \otimes C) \circ (i_C^M \otimes ((h \otimes C) \circ \delta_C)). \end{aligned}$$

In the last computations, the first equality follows from the weak entwined module condition for M , the second one by the properties of i_C^M , the third one by the A -module structure of M and finally, in the fourth one we use the weak entwined module condition of A and the properties of h .

Then, if $\omega'_M \circ \omega_M = id_{M_C \otimes C}$, composing with $M_C \otimes \varepsilon_C$ in the equality

$$\omega'_M \circ \omega_M = ((p_C^M \circ \phi_M) \otimes C) \circ (i_C^M \otimes ((h \otimes C) \circ \delta_C))$$

we obtain that $M_C \otimes \varepsilon_C = p_C^M \circ \omega_M$. Conversely, if $M_C \otimes \varepsilon_C = p_C^M \circ \omega_M$ we have

$$\omega'_M \circ \omega_M = (p_C^M \otimes C) \circ \rho_M \circ \omega_M = ((p_C^M \circ \omega_M) \otimes C) \circ (M_C \otimes \delta_C) = id_{M_C \otimes C}.$$

In the particular instance, $A = M$ it is easy to show that if $\omega'_A \circ \omega_A = id_{A_C \otimes C}$ then $h \wedge h^{-1} = \varepsilon_C \otimes \eta_A$. On the other hand, if $h \wedge h^{-1} = \varepsilon_C \otimes \eta_A$, by the usual arguments, the following equalities hold:

$$\begin{aligned} i_C^A \circ p_C^A \circ \omega_A &= \mu_A \circ (A \otimes h^{-1}) \circ \rho_A \circ \mu_A \circ (i_C^A \otimes h) \\ &= \mu_A \circ (\mu_A \otimes h^{-1}) \circ (A \otimes \psi) \circ ((\rho_A \circ i_C^A) \otimes h) \\ &= \mu_A \circ (\mu_A \otimes h^{-1}) \circ (\mu_A \otimes \psi) \circ (i_C^A \otimes (\rho_A \circ \eta_A) \otimes h) \\ &= \mu_A \circ (\mu_A \otimes h^{-1}) \circ (i_C^A \otimes (\rho_A \circ h)) \\ &= \mu_A \circ (\mu_A \otimes A) \circ (i_C^A \otimes ((h \otimes h^{-1}) \circ \delta_C)) \\ &= \mu_A \circ (i_C^A \otimes (h \wedge h^{-1})) \\ &= i_A^C \otimes \varepsilon_C. \end{aligned}$$

Therefore, $p_C^A \circ \omega_A = A_C \otimes \varepsilon_C$ and then $\omega'_A \circ \omega_A = id_{A_C \otimes C}$.

Finally, let $g \in Hom_C(C, A)$ verifying $g \wedge h = e_{RR}$ and $\psi \circ (C \otimes g) \circ \delta_C = \zeta_A \circ (e_{RR} \wedge g)$. Then, $h \wedge g = \varepsilon_C \otimes \eta_A$ and $g = g \wedge h \wedge g = e_{RR} \wedge g = h^{-1} \wedge h \wedge g = h^{-1}$. \square

Example 1.20. Let H, B be weak Hopf algebras in \mathcal{C} . Let $g: B \rightarrow H$ and $f: H \rightarrow B$ be morphisms of weak Hopf algebras such that $g \circ f = id_H$. As a consequence of 1.19 we obtain that ω_B is an isomorphism if and only if H is a Hopf algebra. This result was proved in Proposition 2.10 of [2].

1.21. Let $A_C \hookrightarrow A$ be a weak C -cleft extension with morphism $h \in Reg^{WR}(C, A)$ and let $M \in \mathcal{M}_A^C(\psi)$. The morphism $\Omega_M = \omega'_M \circ \omega_M$ is an idempotent and then we have a commutative diagram

$$\begin{array}{ccc}
 & M & \\
 \omega_M \nearrow & & \searrow \omega'_M \\
 M_C \otimes C & \xrightarrow{\Omega_M} & M_C \otimes C \\
 r_M \searrow & & \nearrow s_M \\
 & M_C \times C &
 \end{array}$$

where $r_M \circ s_M = id_{M_C \times C}$. Therefore, the morphism $b_M = r_M \circ \omega'_M$ is an isomorphism of right C -comodules with inverse $b_M^{-1} = \omega_M \circ s_M$. The comodule structure of $M_C \times C$ is the one induced by the isomorphism b_M and it is equal to

$$\rho_{M_C \times C} = (r_M \otimes C) \circ (M_C \otimes \delta_C) \circ s_M,$$

because $(\omega_M \otimes C) \circ (M_C \otimes \delta_C) \circ s_M = \rho_M \circ \omega_M \circ s_M$. In the particular case $A = M$ we have that b_A is an isomorphism of algebras where the algebra structure is the one induced by b_A :

$$\eta_{A_C \times C} = b_A \circ \eta_A, \quad \mu_{A_C \times C} = b_A \circ \mu_A \circ (b_A^{-1} \otimes b_A^{-1}).$$

In the next proposition we obtain that $\mu_{A_C \times C}$ can be identified in other way.

Proposition 1.22. Let $A_C \hookrightarrow A$ be a weak C -cleft extension with morphism $h \in Reg^{WR}(C, A)$. Then $\mu_{A_C \times C} = \mu_{A_C \#_{\pi_A}^{x_A} C}$ where

$$\mu_{A_C \#_{\pi_A}^{x_A} C} = r_A \circ (\mu_{A_C} \otimes C) \circ (\mu_{A_C} \otimes \pi_A) \circ (A_C \otimes \chi_A \otimes C) \circ (s_A \otimes s_A)$$

and

$$\pi_A = (\varphi'_A \otimes C) \circ (C \otimes \psi) \circ (\delta_C \otimes h), \quad \chi_A = (\varphi'_A \otimes C) \circ (C \otimes \psi) \circ (\delta_C \otimes i_C^A).$$

Proof. Using the equalities:

$$(A1) \quad \mu_A \circ (\mu_A \otimes \mu_A) \circ (h \otimes A \otimes h^{-1} \otimes h) \circ (C \otimes \psi \otimes C) \circ (\delta_C \otimes \psi) \circ (\delta_C \otimes A) = \mu_A \circ ((h \wedge e_{RR}) \otimes A)$$

and

$$(A2) \quad \mu_A \circ (\mu_A \otimes \mu_A) \circ (h \otimes A \otimes e_{RR} \otimes e_{RR}) \circ (C \otimes \psi \otimes C) \circ (\delta_C \otimes \psi) \circ (\delta_C \otimes i_C^A) = \mu_A \circ (h \otimes i_C^A)$$

we obtain

$$\begin{aligned} & b_A^{-1} \circ \mu_{A_C \#_{\pi_A^A}^{\chi_A} C} \\ &= \mu_A \circ (\mu_A \otimes \mu_A) \circ (A \otimes i_C^A \otimes i_C^A \otimes h) \circ (A \otimes \varphi'_A \otimes \varphi'_A \otimes C) \\ & \quad \circ (A \otimes C \otimes A \otimes C \otimes \psi) \circ (A \otimes C \otimes \psi \otimes h) \circ (i_C^A \otimes \delta_C \otimes i_C^A \otimes C) \circ (s_A \otimes s_A) \\ &= \mu_A \circ ([\mu_A \circ (A \otimes (\mu_A \circ (\mu_A \otimes A) \circ (h \otimes A \otimes h^{-1}) \circ (C \otimes \psi) \circ (\delta_C \otimes A)))]) \\ & \quad \otimes [\mu_A \circ (\mu_A \otimes \mu_A) \circ (h \otimes A \otimes h^{-1} \otimes h) \circ (C \otimes \psi \otimes C) \circ (\delta_C \otimes \psi) \circ (\delta_C \otimes A)]] \\ & \quad \circ (A \otimes C \otimes \psi \otimes A) \circ (i_C^A \otimes \delta_C \otimes i_C^A \otimes h) \circ (s_A \otimes s_A) \\ &= \mu_A \circ ([\mu_A \circ (A \otimes (\mu_A \circ (\mu_A \otimes A) \circ (h \otimes A \otimes h^{-1}) \circ (C \otimes \psi) \circ (\delta_C \otimes A)))]) \\ & \quad \otimes [\mu_A \circ ((h \wedge e_{RR}) \otimes A)]) \circ (A \otimes C \otimes \psi \otimes A) \circ (i_C^A \otimes \delta_C \otimes i_C^A \otimes h) \circ (s_A \otimes s_A) \\ &= \mu_A \circ (\mu_A \otimes A) \circ (\mu_A \otimes (e_{RR} \wedge e_{RR}) \otimes A) \circ (A \otimes \mu_A \otimes C \otimes A) \\ & \quad \circ (i_C^A \otimes h \otimes \psi \otimes h) \circ (A_C \otimes \delta_C \otimes i_C^A \otimes C) \circ (s_A \otimes s_A) \\ &= \mu_A \circ (\mu_A \otimes A) \circ (i_C^A \otimes [\mu_A \circ (\mu_A \otimes \mu_A) \circ (h \otimes A \otimes e_{RR} \otimes e_{RR}) \\ & \quad \circ (C \otimes \psi \otimes C) \circ (\delta_C \otimes \psi) \circ (\delta_C \otimes i_C^A)]) \otimes h) \circ (s_A \otimes s_A) \\ &= \mu_A \circ (\mu_A \otimes A) \circ (i_C^A \otimes [\mu_A \circ (h \otimes i_C^A)]) \otimes h) \circ (s_A \otimes s_A) \\ &= \mu_A \circ (b_A^{-1} \otimes b_A^{-1}). \end{aligned}$$

In the last computations, the first and the second equalities follows by definition, the third one by the equality (A1), the fourth and the fifth ones by the properties of ψ , the sixth one by (A2) and finally, the seventh one is a trivial calculus.

Therefore, $\mu_{A_C \times C} = \mu_{A_C \#_{\pi_A^A}^{\chi_A} C}$. \square

Examples 1.23. (i) If we work with entwined structures, the last result is Proposition 2.11 of [20]. In this context $\times = \otimes$ and the algebra $A_C \#_{\pi_A^A}^{\chi_A} C$, called the cross product algebra, was studied by Brzeziński in [13].

(ii) Let H, B be weak Hopf algebras in a symmetric monoidal category \mathcal{C} with split idempotents. Let $g: B \rightarrow H$ be a morphism of weak Hopf algebras and $f: H \rightarrow B$ be a morphism of coalgebras such that $g \circ f = id_H$ and $f \circ \eta_H = \eta_B$. Under these conditions it is possible to prove, using similar computations to the ones developed in 4.2 of [3], that the morphism $\Omega_B = \omega'_B \circ \omega_B$ admits the following new formulation:

$$\Omega_B = (\varphi_{B_H} \otimes \mu_H) \circ (H \otimes c_{H, B_H} \otimes H) \circ ((\delta_H \circ \eta_H) \otimes B_H \otimes H).$$

Therefore, the object $B_H \times H$ is the tensor product of B_H and H in the representation category of H . This category is denoted by $\text{Rep}(H)$ and were studied in [12] and [26] (see also [25]).

Moreover, if σ_{B_H} is the morphism obtained in 1.17, we can define the following morphisms

$$\begin{aligned}\eta_{B_H \sharp_{\sigma_{B_H}} H} : K &\rightarrow B_H \times H, & \mu_{B_H \sharp_{\sigma_{B_H}} H} : B_H \times H \otimes B_H \times H &\rightarrow B_H \times H, \\ \rho_{B_H \sharp_{\sigma_{B_H}} H} : B_H &\rightarrow B_H \times H \otimes H\end{aligned}$$

by

$$\begin{aligned}\eta_{B_H \sharp_{\sigma_{B_H}} H} &= r_B \circ (\eta_{B_H} \otimes \eta_H), \\ \mu_{B_H \sharp_{\sigma_{B_H}} H} &= r_B \circ (\mu_{B_H} \otimes H) \circ (\mu_{B_H} \otimes \sigma_{B_H} \otimes \mu_H) \circ (B_H \otimes \varphi_{B_H} \otimes \delta_{H \otimes H}) \\ &\quad \circ (B_H \otimes H \otimes c_{H, B_H} \otimes H) \circ (B_H \otimes \delta_H \otimes B_H \otimes H) \circ (s_B \otimes s_B), \\ \rho_{B_H \sharp_{\sigma_{B_H}} H} &= (r_B \otimes H) \circ (B_H \otimes \delta_H) \circ s_B.\end{aligned}$$

If we denote by $B_H \sharp_{\sigma_{B_H}} H$ (the crossed product of B_H and H) the triple

$$(B_H \times H, \eta_{B_H \sharp_{\sigma_{B_H}} H}, \mu_{B_H \sharp_{\sigma_{B_H}} H}),$$

then $B_H \sharp_{\sigma_{B_H}} H$ is an algebra, $(B_H \times H, \rho_{B_H \sharp_{\sigma_{B_H}} H})$ is a right H -comodule and the morphism $b_B : B \rightarrow B_H \sharp_{\sigma_{B_H}} H$ is an isomorphism of algebras and right H -comodules (see Theorem 2.8 [3]). Theorem 2.8 of [3] is the weak version of the result obtained by Blattner, Cohen and Montgomery in [9]. Moreover, in the Hopf algebra case, if f is an algebra morphism, we have $\sigma_{B_H} = \varepsilon_H \otimes \varepsilon_H \otimes \eta_{B_H}$ and then $B_H \sharp_{\sigma_{B_H}} H$ is the smash product of B_H and H , denoted by $B_H \sharp H$. Observe that the product of $B_H \sharp H$ is

$$\mu_{B_H \sharp H} = (\mu_{B_H} \otimes \mu_H) \circ (B_H \otimes ((\varphi_{B_H} \otimes H) \circ (H \otimes c_{H, B_H}) \circ (\delta_H \otimes B_H)) \otimes H).$$

In the weak Hopf algebra case, if f is a morphism of algebras, $\sigma_{B_H} = p_H^B \circ \Pi_L^B \circ f \circ \mu_H$ and then

$$\begin{aligned}\mu_{B_H \sharp_{\sigma_{B_H}} H} &= r_B \circ (\mu_{B_H} \otimes \mu_H) \\ &\quad \circ (B_H \otimes ((\varphi_{B_H} \otimes H) \circ (H \otimes c_{H, B_H}) \circ (\delta_H \otimes B_H)) \otimes H) \circ (s_B \otimes s_B).\end{aligned}$$

As a consequence, for analogy with the Hopf algebra case, when $\sigma_{B_H} = p_H^B \circ \Pi_L^B \circ f \circ \mu_H$, we will denote the triple $B_H \sharp_{\sigma_{B_H}} H$ by $B_H \sharp H$ (the smash product of B_H and H).

Finally, Proposition 1.22 implies that the product $\mu_{B_H \sharp_{\sigma_{B_H}} H}$ is equal to $\mu_{B_H \sharp_{\pi_B}^{\chi_B} H}$ where

$$\mu_{B_H \sharp_{\pi_B}^{\chi_B} H} = r_B \circ (\mu_{B_H} \otimes H) \circ (\mu_{B_H} \otimes \pi_B) \circ (B_H \otimes \chi_B \otimes H) \circ (s_B \otimes s_B)$$

and

$$\pi_B = (\varphi'_B \otimes H) \circ (H \otimes \psi) \circ (\delta_H \otimes f), \quad \chi_B = (\varphi'_B \otimes H) \circ (H \otimes \psi) \circ (\delta_H \otimes i_H^B).$$

2. Weak cocleft coextensions

In this section we study the dual results of the previous one. If we particularize these results to the case of entwining structures we obtain the theory developed in Section 3 of [20]. In this section, the arguments and computations are similar to the ones used in Section 1, but passing to the opposite category, and then we leave the details to the reader.

Proposition 2.1. *Let (A, C, ψ) be a weak entwining structure such that there exists a action ϕ_C verifying that (C, ϕ_C, δ_C) belongs to $\mathcal{M}_A^C(\psi)$. If for all $(M, \phi_M, \rho_M) \in \mathcal{M}_A^C(\psi)$, we denote by M^A the coequalizer of ϕ_M and $\beta_M = (M \otimes (\varepsilon_C \circ \phi_C)) \circ (\rho_M \otimes A)$ and by l_A^M the projection of M on M^A , we have the following:*

- (i) *The triple $(C^A, \varepsilon_{C^A}, \delta_{C^A})$ is a coalgebra in \mathcal{C} , where $\varepsilon_{C^A} : C^A \rightarrow C^A \otimes C^A$ and $\delta_{C^A} : C^A \rightarrow C^A \otimes C^A$ are the factorizations of ε_C and $(l_A^C \otimes l_A^C) \circ \delta_C$ respectively, through the coequalizer l_A^C .*
- (ii) *The pair (M^A, ρ_{M^A}) is a right C^A -comodule, where $\rho_{M^A} : M_C \rightarrow M_C \otimes C^A$ is the factorization of $(l_A^M \otimes l_A^C) \circ \rho_M$ through the coequalizer l_A^M .*

Example 2.2 (See Section 3 of [3] for more details). Let H, B be weak Hopf algebras in a symmetric monoidal category \mathcal{C} with split idempotents. Let $j : H \rightarrow B$ be a morphism of weak Hopf algebras and $t : B \rightarrow H$ be a morphism of algebras such that $t \circ j = id_H$ and $\varepsilon_H \circ t = \varepsilon_B$. If we define $\phi_B : B \otimes H \rightarrow B$ and the intertwining $\psi : B \otimes H \rightarrow H \otimes B$ by

$$\phi_B = \mu_B \circ (B \otimes j), \quad \psi = (H \otimes \phi_B) \circ (c_{B,H} \otimes H) \circ (B \otimes \delta_H)$$

we have that (H, B, ψ) is a weak entwining structure where $e_{RR} = t \circ \Pi_B^R$. Also, (B, ϕ_B, δ_B) belongs to the category $\mathcal{M}_H^B(\psi)$.

The morphism $k_H^B : B \rightarrow B$ defined by

$$k_H^B = \phi_B \circ (B \otimes (t \circ \lambda_B)) \circ \delta_B$$

is idempotent in \mathcal{C} and, as a consequence, we obtain that there exist an epimorphism l_H^B , a monomorphism n_H^B and an object B^H such that the diagram

$$\begin{array}{ccc} B & \xrightarrow{k_H^B} & B \\ & \searrow l_H^B & \nearrow n_H^B \\ & B^H & \end{array}$$

commutes and $l_H^B \circ n_H^B = id_{B^H}$. Moreover, the next diagrams are coequalizer diagrams in \mathcal{C}

$$\begin{array}{ccccc} B \otimes H & \xrightarrow{\phi_B} & B & \xrightarrow{l_H^B} & B^H, \\ & \xrightarrow{\phi_B \circ (B \otimes \Pi_H^L)} & & & \\ \\ B \otimes H & \xrightarrow{\phi_B} & B & \xrightarrow{l_H^B} & B^H, \\ & \xrightarrow{\phi_B \circ (B \otimes \overline{\Pi}_H^L)} & & & \end{array}$$

Therefore, the object defined by the equalizer of ϕ_B and β_B is the same that the one defined by the equalizer of ϕ_B and $\phi_B \circ (B \otimes \overline{\Pi}_H^L)$ because, in this situation, $\phi_B \circ (B \otimes \overline{\Pi}_H^L) = \beta_B$.

Remark 2.3. Let (A, C, ψ) be a weak entwining structure. If there exists an action ϕ_C verifying that $(C, \phi_C, \delta_C) \in \mathcal{M}_A^C(\psi)$ and $h' \in Hom_C(C, A)$ is a morphism of right A -modules, we have that $h' \wedge e_{RR} = h'$.

Definition 2.4. Let (A, C, ψ) be a weak entwined structure and suppose that $(C, \phi_C, \delta_C) \in \mathcal{M}_A^C(\psi)$. We will say that $C \twoheadrightarrow C^A$ is a weak A -cocleft coextension if there exists a morphism $h' \in Reg^{WR}(C, A)$ of right A -modules such that

$$\mu_A \circ (A \otimes h'^{-1}) \circ \psi = (e_{RR} \wedge h'^{-1}) \circ \beta_C$$

where $\beta_C = (C \otimes (\varepsilon_C \circ \phi_C)) \circ (\delta_C \otimes A)$ is the morphism defined in 2.1. Also, as in 1.13, we can suppose without loss of generality that $e_{RR} \wedge h'^{-1} = h'^{-1}$.

Remarks 2.5. (i) Let (A, C, ψ) be a weak entwining structure. If there exists an action ϕ_C verifying that $(C, \phi_C, \delta_C) \in \mathcal{M}_A^C(\psi)$ and $h' \in Hom_C(C, A)$ is a morphism of right A -modules such that h' in $Reg^{WR}(C, A)$, the intertwining ψ is completely determined in the following form:

$$\psi = ((\mu_A \circ (h'^{-1} \otimes h')) \otimes C) \circ (C \otimes (\delta_C \circ \phi_C)) \circ (\delta_C \otimes A).$$

(ii) Let (A, C, ψ) be a entwining structure and suppose that $(C, \phi_C, \delta_C) \in \mathcal{M}_A^C(\psi)$. If $h' \in Reg(C, A)$ is a morphism of right A -modules we have

$$\mu_A \circ (A \otimes h'^{-1}) \circ \psi = h'^{-1} \circ \beta_C = (e_{RR} \wedge h'^{-1}) \circ \beta_C$$

and then a cocleft coextension in the sense of [20] is a weak cocleft coextension.

Example 2.6. Let B and H be weak Hopf algebras in the same conditions of 2.2, then $t \in Reg^{WR}(B, H)$ with inverse $t^{-1} = t \circ \lambda_B$ and $B = (B, \phi_B, \delta_B) \twoheadrightarrow B^H$ is a weak H -cocleft coextension. When $H = B$ and $f = g = id_H$ we have the trivial example of weak H -cocleft coextension.

Proposition 2.7. Let (A, C, ψ) be a weak entwining structure such that there exists an action ϕ_C verifying that $(C, \phi_C, \delta_C) \in \mathcal{M}_A^C(\psi)$. Let $h' : C \rightarrow A$ be a morphism of right A -modules in $\text{Reg}^{WR}(C, A)$. The following are equivalent:

- (i) $C \rightrightarrows C^A$ is a weak A -cocleft coextension.
- (ii) For all object M in $\mathcal{M}_A^C(\psi)$ the morphism $k_A^M = \phi_M \circ (M \otimes h'^{-1}) \circ \rho_M : M \rightarrow M$ factors through the coequalizer l_A^M , i.e., there exists a morphism $n_A^M : M^A \rightarrow M$ such that $n_A^M \circ l_A^M = k_A^M$.

Example 2.8. Let H, B be weak Hopf algebras in a symmetric monoidal category with split idempotents \mathcal{C} . Let $j : H \rightarrow B$ be a morphism of weak Hopf algebras and $t : B \rightarrow H$ be a morphism of algebras such that $t \circ j = id_H$ and $\varepsilon_H \circ t = \varepsilon_B$. In these conditions, for $M = B$, the morphism k_H^B is the one defined in Example 2.2.

Proposition 2.9. Let $C \rightrightarrows C^A$ be a weak A -cocleft coextension with $h' : C \rightarrow A$ the morphism of right A -modules in $\text{Reg}^{WR}(C, A)$. Then the morphism $r_C : C \rightarrow A \otimes C$ defined by

$$r_C = (\mu_A \otimes C) \circ (h' \otimes \psi) \circ (\delta_C \otimes h'^{-1}) \circ \delta_C$$

factors through the coequalizer l_A^C . Moreover, if r'_C is the factorization of r_C , we have the following equality:

$$(\mu_A \otimes C \otimes C) \circ (A \otimes \psi \otimes C) \circ (r'_C \otimes r'_C) \circ \delta_{C^A} = (A \otimes \delta_C) \circ r'_C.$$

Finally, if we define the morphism $r_{C^A} : C^A \rightarrow A \otimes C^A$ by $r_{C^A} = (A \otimes l_A^C) \circ r'_C$ we obtain:

$$(\mu_A \otimes l_A^C \otimes C^A) \circ (A \otimes \psi \otimes C^A) \circ (r'_C \otimes r_{C^A}) \circ \delta_{C^A} = (A \otimes \delta_{C^A}) \circ r_{C^A}.$$

Examples 2.10. (i) In Proposition 3.5 of [20] we can find a similar result in the context of A -cocleft coextensions for entwining structures.

(ii) Let H, B be weak Hopf algebras in a symmetric monoidal category with split idempotents \mathcal{C} . Let $j : H \rightarrow B$ be a morphism of weak Hopf algebras and $t : B \rightarrow H$ be a morphism of algebras such that $t \circ j = id_H$ and $\varepsilon_H \circ t = \varepsilon_B$. In these conditions, r_{BH} is the morphism defined in Section 3 of [3]. The morphism r_{BH} verifies

- (1) $(\varepsilon_H \otimes B^H) \circ r_{BH} = id_{BH}$.
- (2) $(H \otimes \varepsilon_{BH}) \circ r_{BH} = (\Pi_H^L \otimes \varepsilon_{BH}) \circ r_{BH}$.
- (3) $(H \otimes \varepsilon_{BH} \otimes B^H) \circ (r_{BH} \otimes B^H) \circ \delta_{BH} = (\Pi_H^L \otimes B^H) \circ r_{BH}$.
- (4) $(H \otimes \delta_{BH}) \circ r_{BH} = (\mu_H \otimes B^H \otimes B^H) \circ (H \otimes c_{BH, H} \otimes B^H) \circ (r_{BH} \otimes r_{BH}) \circ \delta_{BH}$.
- (5) $((H \otimes \varepsilon_{BH}) \circ r_{BH}) \otimes B^H \circ c_{BH, BH} \circ \delta_{BH} = (\overline{\Pi}_H^L \otimes B^H) \circ r_{BH}$.

By 2.9 we can add to the last equalities the new property

$$(6) (\mu_H \otimes l_H^B \otimes B^H) \circ (H \otimes \psi \otimes B^H) \circ (r'_B \otimes r_{B^H}) \circ \delta_{B^H} = (H \otimes \delta_{B^H}) \circ r_{B^H},$$

where ψ is the entwining defined in 2.2.

Moreover, if t is a morphism of coalgebras (B_H, r_{B^H}) is a left H -comodule (see Proposition 2.5 of [2]).

Proposition 2.11. *Let $C \rightrightarrows C^A$ be a weak A -cocleft coextension with $h' : C \rightarrow A$ the morphism of right A -modules in $\text{Reg}^{WR}(C, A)$. Then the morphism $\gamma_C : C \rightarrow A \otimes A$ defined by*

$$\gamma_C = (\mu_A \otimes h') \circ (h' \otimes \psi) \circ (\delta_C \otimes h'^{-1}) \circ \delta_C$$

factors through the coequalizer l_A^C . Moreover, if γ_{C^A} is the factorization of γ_C , then

$$\gamma_{C^A} = (h' \otimes h') \circ \delta_C \circ n_A^C.$$

2.12. Let $C \rightrightarrows C^A$ be a weak A -cocleft coextension with morphism $h' \in \text{Reg}^{WR}(C, A)$ and let $M \in \mathcal{M}_A^C(\psi)$. The morphisms

$$\varpi_M : M^A \otimes A \rightarrow M, \quad \varpi'_M : M \rightarrow M^A \otimes A$$

defined by $\varpi_M = \phi_M \circ (n_A^M \otimes A)$ and $\varpi'_M = (l_A^M \otimes h') \circ \rho_M$ verify the equality $\varpi_M \circ \varpi'_M = id_M$ because we have $\varpi_M \circ \varpi'_M = \phi_M \circ (M \otimes e_{RR}) \circ \rho_M = id_M$. Also, the equality $\varpi'_M \circ \varpi_M = id_{M^A \otimes A}$ it is not true in general and the dual version of Proposition 1.19 is the following:

Proposition 2.13. *Let $C \rightrightarrows C^A$ be a weak A -cocleft coextension with $h' : C \rightarrow A$ the morphism of right A -modules in $\text{Reg}^{WR}(C, A)$ and let $M \in \mathcal{M}_A^C(\psi)$. Then, $\varpi'_M \circ \varpi_M = id_{M^A \otimes A}$ if and only if $M^A \otimes \eta_A = \varpi'_M \circ n_A^M$.*

As a consequence, if $M = C$ we have $\varpi'_C \circ \varpi_C = id_{C^A \otimes A}$ if and only if $h' \wedge h'^{-1} = \varepsilon_C \otimes \eta_A$. Therefore, if $\varpi'_C \circ \varpi_C = id_{C^A \otimes A}$, the left weak inverse of h' is unique.

2.14. Let $C \rightrightarrows C^A$ be a weak A -cocleft coextension with morphism $h' \in \text{Reg}^{WR}(C, A)$ and let $M \in \mathcal{M}_A^C(\psi)$. The morphism $\gamma_M = \varpi'_M \circ \varpi_M$ is an idempotent and then we have a commutative diagram

$$\begin{array}{ccc}
 & M & \\
 \varpi_M \nearrow & & \searrow \varpi'_M \\
 M^A \otimes A & \xrightarrow{\gamma_M} & M^A \otimes A \\
 u_M \searrow & & \nearrow v_M \\
 & M^A \boxtimes A &
 \end{array}$$

where $u_M \circ v_M = id_{M^A \square A}$. Therefore, the morphism $d_M = u_M \circ \varpi'_M$ is an isomorphism of right A -modules with inverse $d_M^{-1} = \varpi_M \circ v_M$. The module structure of $M^A \square A$ is the one induced by the isomorphism d_M and it is equal to

$$\phi_{M^A \square A} = u_M \circ (M^A \otimes \mu_A) \circ (\varpi'_M \otimes \mu_A) \circ (n_A^M \otimes A \otimes A) \circ (v_M \otimes A).$$

In the particular case $C = M$ we have that d_C is an isomorphism of coalgebras where the coalgebra structure is the one induced by d_C :

$$\varepsilon_{C^A \square A} = \varepsilon_C \circ d_C^{-1}, \quad \delta_{C^A \square A} = (d_C \otimes d_C) \circ \delta_C \circ d_C^{-1}.$$

In the following proposition we obtain that $\delta_{C^A \square A}$ can be identified using a crossed coproduct.

Proposition 2.15. *Let $C \twoheadrightarrow C^A$ be a weak A -cocleft coextension with morphism $h' \in \text{Reg}^{WR}(C, A)$. Then $\delta_{C^A \square A} = \delta_{C^A \ominus_{\varsigma_C}^{\theta_C} A}$ where*

$$\delta_{C^A \ominus_{\varsigma_C}^{\theta_C} A} = (u_C \otimes u_C) \circ (C^A \otimes \theta_C \otimes A) \circ (\delta_{C^A} \otimes \varsigma_C) \otimes (\delta_{C^A} \otimes A) \circ v_C$$

and

$$\varsigma_C = (\mu_A \otimes h') \circ (A \otimes \psi) \circ (r'_C \otimes A), \quad \theta_C = (\mu_A \otimes l_A^C) \circ (A \otimes \psi) \circ (r'_C \otimes A).$$

Examples 2.16. (i) In the entwined case, the last result is Proposition 3.6 of [20]. In this context $\otimes = \square$ and the coalgebra $C^A \ominus_{\varsigma_C}^{\theta_C} A$, called the cross coproduct coalgebra, is the dual of the one studied by Brzeziński in [13].

(ii) Let H, B be weak Hopf algebras in a symmetric monoidal category with split idempotents. Let $j: H \rightarrow B$ be a morphism of weak Hopf algebras and $t: B \rightarrow H$ be a morphism of algebras such that $t \circ j = id_H$ and $\varepsilon_H \circ t = \varepsilon_B$. Let γ_{BH} be the morphism defined in 2.11 and put

$$\begin{aligned} \varepsilon_{B^H \ominus_{\gamma_{BH}} H} : B^H \square H &\rightarrow K, & \delta_{B^H \ominus_{\gamma_{BH}} H} : B^H \square H &\rightarrow B^H \square H \otimes B^H \square H, \\ \psi_{B^H \ominus_{\gamma_{BH}} H} : B^H \square H \otimes H &\rightarrow B^H \square H \end{aligned}$$

where

$$\begin{aligned} \varepsilon_{B^H \ominus_{\gamma_{BH}} H} &= (\varepsilon_{B^H} \otimes \varepsilon_H) \circ v_B, \\ \delta_{B^H \ominus_{\gamma_{BH}} H} &= (u_B \otimes u_B) \circ (B^H \otimes \mu_H \otimes B^H \otimes H) \circ (B^H \otimes H \otimes c_{B^H, H} \otimes H) \\ &\quad \circ (B^H \otimes r_{BH} \otimes \mu_{H \otimes H}) \circ (\delta_{B^H} \otimes \gamma_{BH} \otimes \delta_H) \circ (\delta_{B^H} \otimes H) \circ v_B, \end{aligned}$$

Then, if we denote by $B^H \ominus_{\gamma_{BH}} H$ (the crossed coproduct of B^H and H) the triple

$$(B^H \boxtimes H, \varepsilon_{B^H \ominus_{\gamma_{BH}} H}, \delta_{B^H \ominus_{\gamma_{BH}} H}),$$

we have that $B^H \ominus_{\gamma_{BH}} H$ is a coalgebra and $d_B : B \rightarrow B^H \ominus_{\gamma_{BH}} H$ is an isomorphism of coalgebras and right H -modules.

Proposition 2.15 implies that the coproduct $\delta_{B^H \ominus_{\gamma_{BH}} H}$ is equal to $\delta_{B^H \ominus_{\varepsilon_B} H}$ where

$$\delta_{B^H \ominus_{\varepsilon_B} H} = (u_B \otimes u_B) \circ (B^H \otimes \theta_B \otimes H) \circ (\delta_{B^H} \otimes \varsigma_B) \otimes (\delta_{B^H} \otimes H) \circ v_B$$

and

$$\varsigma_B = (\mu_H \otimes t) \circ (H \otimes \psi) \circ (r'_B \otimes H), \quad \theta_B = (\mu_H \otimes l_H^B) \circ (H \otimes \psi) \circ (r'_B \otimes H).$$

In the Hopf algebra case (H and B Hopf algebras) this result is the dual of the one obtained by Blattner, Cohen and Montgomery. In this case, if t is a coalgebra morphism, we have $\gamma_{BH} = \varepsilon_{B^H} \otimes \eta_H \otimes \eta_H$ and then $B^H \ominus_{\gamma_{BH}} H$ is the smash coproduct of B^H and H , denoted by $B^H \ominus H$. In $B^H \ominus H$ the coproduct is

$$\delta_{B^H \ominus H} = (B^H \otimes ((\mu_H \otimes B^H) \circ (H \otimes c_{B^H, H}) \circ (r_{B^H} \otimes H)) \otimes H) \circ (\delta_{B^H} \otimes \delta_H).$$

Finally, when t is a morphism of weak Hopf algebras we have $\gamma_{BH} = \delta_H \circ \Pi_H^L \circ t \circ n_H^B$ and then the expression of $\delta_{B^H \ominus_{\gamma_{BH}} H}$ is:

$$\begin{aligned} \delta_{B^H \ominus_{\gamma_{BH}} H} &= (u_B \otimes u_B) \circ (B^H \otimes ((\mu_H \otimes B^H) \circ (H \otimes c_{B^H, H}) \circ (r_{B^H} \otimes H)) \otimes H) \\ &\quad \circ (\delta_{B^H} \otimes \delta_H) \circ v_B. \end{aligned}$$

As a consequence, for analogy with the Hopf algebra case, when $\gamma_{BH} = \delta_H \circ \Pi_H^L \circ t \circ n_H^B$, we will denote the triple $B^H \ominus_{\gamma_{BH}} H$ by $B^H \ominus H$ (the smash coproduct of B^H and H).

3. The mixed case

3.1. Let A and C be algebras coalgebras in \mathcal{C} . Let $g : A \rightarrow C$ and $f : C \rightarrow A$ be morphisms of algebras and coalgebras such that

- (i) $f \circ g = id_C$.
- (ii) There exists a weak entwined structure (A, C, ψ) in \mathcal{C} such that $(A, \mu_A, \rho_A = (A \otimes g) \circ \delta_A) \in \mathcal{M}_A^C(\psi)$.
- (iii) There exists a weak entwined structure (C, A, ψ') in \mathcal{C} such that $(A, \phi_A = \mu_A \circ (A \otimes f), \delta_A) \in \mathcal{M}_C^A(\psi')$.
- (iv) $f \in Reg^{WR}(C, A)$, $g \in Reg^{WR}(A, C)$ and

$$\begin{aligned}\psi \circ (C \otimes f^{-1}) \circ \delta_C &= \zeta_A \circ (e_{RR} \wedge f^{-1}), \\ \mu_A \circ (A \otimes g^{-1}) \circ \psi' &= (e_{RR} \wedge g^{-1}) \circ \beta_C.\end{aligned}$$

Under these conditions, $A_C \hookrightarrow A$ is a weak C -cleft extension with morphism $f \in \text{Reg}^{WR}(C, A)$ and $A \twoheadrightarrow A^C$ is a weak C -cocleft coextension with morphism $g \in \text{Reg}^{WR}(A, C)$. Therefore, there exist two isomorphisms, defined in 1.21 and 2.14, $b_A: A \rightarrow A_C \times C$, $d_A: A \rightarrow A^C \boxtimes C$, and, as a consequence, $d_A \circ b_A^{-1}: A_C \times C \rightarrow A^C \boxtimes C$ is an isomorphism.

Moreover, if $f \circ g^{-1} = f^{-1} \circ g$ we obtain that $q_C^A = k_C^A$ and then, there exists a unique morphism $y'_A: A^C \rightarrow A_C$ such that $y'_A \circ l_C^A = p_C^A$ and $i_C^A \circ y'_A = n_C^A$. Finally, defining $y_A: A^C \boxtimes C \rightarrow A_C \times C$ by $y_A = r_A \circ (y'_A \otimes C) \circ v_A$, we obtain that $(d_A \circ b_A^{-1})^{-1} = y_A$. Indeed:

$$\begin{aligned}d_A \circ b_A^{-1} \circ r_A \circ (y'_A \otimes C) \circ v_A &= d_A \circ \omega_A \circ \omega'_A \circ \omega_A \circ (y'_A \otimes C) \circ v_A \\ &= d_A \circ \mu_A \circ (i_C^A \otimes f) \circ (y'_A \otimes C) \circ v_A \\ &= d_A \circ \mu_A \circ (n_C^A \otimes f) \circ v_A = u_A \circ \varpi'_A \circ \varpi_A \circ v_A \\ &= u_A \circ v_A \circ u_A \circ v_A = id_{A^C \boxtimes C}.\end{aligned}$$

Example 3.2. Let H, B be weak Hopf algebras in a symmetric monoidal category \mathcal{C} with split idempotents. Suppose that $g: B \rightarrow H$ and $f: H \rightarrow B$ are morphisms of weak Hopf algebras such that $g \circ f = id_H$. Then

- (i) The triple (B, H, ψ) is a weak entwining structure where $\psi = (B \otimes \mu_A) \circ (c_{H,B} \otimes H) \circ (H \otimes \rho_B)$ and $\rho_B = (B \otimes g) \circ \delta_B$. Also, $(B, \mu_B, \rho_B) \in \mathcal{M}_B^H(\psi)$.
- (ii) The triple (H, B, ψ') is a weak entwining structure where $\psi' = (H \otimes \phi_B) \circ (c_{B,H} \otimes H) \circ (B \otimes \delta_H)$ and $\phi_B = \mu_B \circ (B \otimes f)$. Also, $(B, \phi_B, \delta_B) \in \mathcal{M}_H^B(\psi')$.

Then, by 3.1, we obtain that $B_H \hookrightarrow B \twoheadrightarrow B^H$ is a weak H -cleft extension with morphism $f \in \text{Reg}^{WR}(H, B)$ and is a weak H -cocleft coextension with morphism $g \in \text{Reg}^{WR}(B, H)$. In this situation $q_H^B = k_H^B$ and the morphism y'_B is an identity, $B_H = B^H$. Thus

$$B_H \xrightarrow{i_H^B} B \xrightleftharpoons[(B \otimes \Pi_H^L) \circ \rho_B]{\rho_B} B \otimes H$$

is an equalizer diagram and

$$B \otimes H \xrightleftharpoons[\phi_B \circ (B \otimes \Pi_H^L)]{\phi_B} B \xrightarrow{p_H^B} B_H$$

is a coequalizer diagram.

Moreover, $\omega_B = \varpi_B$, $\omega'_B = \varpi'_B$ and then $B_H \times H = B^H \boxtimes H$.

The object B_H is an algebra coalgebra and in Proposition 2.8 of [2] we prove that the triple $(B_H, \varphi_{B_H}, r_{B_H})$ belongs to ${}^H_H\mathcal{WYD}$ where ${}^H_H\mathcal{WYD}$ denotes the category of left weak Yetter–Drinfeld modules over H defined in [2].

As a consequence, we have the following theorem, the weak version of Radford’s theorem, proved in [2].

Theorem 3.3. *Let H, B be weak Hopf algebras in symmetric monoidal category with split idempotents \mathcal{C} . Let $g: B \rightarrow H$ and $f: H \rightarrow B$ be morphisms of weak Hopf algebras such that $g \circ f = id_H$. Then there exists an object B_H living in ${}^H_H\mathcal{WYD}$ such that B is isomorphic to $B_H \times H$ as weak Hopf algebras, being the (co)algebra structure in $B_H \times H$ the smash (co)product. The expression for the antipode of $B_H \times H$ is*

$$\begin{aligned} \lambda_{B_H \times H} &= r_B \circ (\varphi_{B_H} \otimes H) \circ (H \otimes c_{H, B_H}) \circ ((\delta_H \circ \lambda_H \circ \mu_H) \otimes \lambda_{B_H}) \\ &\quad \circ (H \otimes c_{B_H, H}) \circ (r_{B_H} \otimes H) \circ s_B. \end{aligned}$$

3.4. As a particular instance of 3.1, we obtain the results of [20] for the entwining context. Also, we recover the central theorem about Hopf algebras with projection and the Majid’s bosonization process in braided monoidal categories. In this case we must to change the weak Yetter–Drinfeld modules by the usual Yetter–Drinfeld modules.

Theorem 3.5. *Let H, B be Hopf algebras in braided monoidal category with split idempotents \mathcal{C} . Let $g: B \rightarrow H$ and $f: H \rightarrow B$ be morphisms of Hopf algebras such that $g \circ f = id_H$. Then there exists an object B_H living in ${}^H_H\mathcal{YD}$ such that B is isomorphic to $B_H \otimes H$ as Hopf algebras, being the (co)algebra structure in $B_H \otimes H$ the smash (co)product. The expression for the antipode of $B_H \otimes H$ is*

$$\begin{aligned} \lambda_{B_H \otimes H} &= (\varphi_{B_H} \otimes H) \circ (H \otimes c_{H, B_H}) \circ ((\delta_H \circ \lambda_H \circ \mu_H) \otimes \lambda_{B_H}) \\ &\quad \circ (H \otimes c_{B_H, H}) \circ (r_{B_H} \otimes H). \end{aligned}$$

Proof. The line of this proof is the one developed in 3.2 but adapted to the braided case, for example see [1]. \square

Acknowledgments

The authors would like to thank the referee for his constructive and interesting comments.

The authors have been supported by Ministerio de Ciencia y Tecnología, by Xunta de Galicia and by FEDER, Projects: BFM2003-07353-C02-01, BFM2003-07353-C02-02, PGIDITO4PXIC32202PN, PGIDITO4PXIC20703PN.

References

- [1] J.N. Alonso Álvarez, J.M. Fernández Vilaboa, Cleft extensions in braided categories, *Comm. Algebra* 28 (7) (2000) 3185–3196.
- [2] J.N. Alonso Álvarez, R. González Rodríguez, J.M. Fernández Vilaboa, M.P. López López, E. Villanueva Novoa, Weak Hopf algebras with projection and weak smash bialgebra structures, *J. Algebra* 269 (2) (2003) 701–725.
- [3] J.N. Alonso Álvarez, R. González Rodríguez, Crossed products for weak Hopf algebras with coalgebra splitting, *J. Algebra* 281 (2) (2004) 731–752.
- [4] J.N. Alonso Álvarez, J.M. Fernández Vilaboa, R. González Rodríguez, A.B. Rodríguez Raposo, Weak C -cleft extensions and weak Galois extensions, preprint, 2004.
- [5] Y. Bespalov, Crossed modules and quantum groups in braided categories, *Appl. Categ. Structures* 5 (2) (1997) 155–204.
- [6] Y. Bespalov, B. Drabant, Cross product bialgebras I, *J. Algebra* 219 (2) (1999) 466–505.
- [7] Y. Bespalov, B. Drabant, Cross product bialgebras II, *J. Algebra* 240 (2) (2001) 445–504.
- [8] Y. Bespalov, B. Drabant, Hopf (bi-)modules and crossed modules in braided monoidal categories, *J. Pure Appl. Algebra* 123 (1998) 105–129.
- [9] R. Blattner, M. Cohen, S. Montgomery, Crossed products and inner actions of Hopf algebras, *Trans. Amer. Math. Soc.* 298 (2) (1986) 671–711.
- [10] G. Böhm, Doi–Hopf modules over weak Hopf algebras, *Comm. Algebra* 28 (2000) 4687–4698.
- [11] G. Böhm, F. Nill, K. Szlachányi, Weak Hopf algebras, I: Integral theory and C^* -structure, *J. Algebra* 221 (2) (1999) 385–438.
- [12] G. Böhm, K. Szlachányi, Weak Hopf algebras, II: Representation theory, dimensions and the Markov trace, *J. Algebra* 233 (2000) 156–212.
- [13] T. Brzeziński, Crossed products by a coalgebra, *Comm. Algebra* 25 (11) (1997) 3551–3575.
- [14] T. Brzeziński, On modules associated to coalgebra Galois extensions, *J. Algebra* 215 (1999) 290–317.
- [15] T. Brzeziński, S. Majid, Coalgebra bundles, *Comm. Math. Phys.* 191 (1998) 467–492.
- [16] S. Caenepeel, E. de Groot, Modules over weak entwining structures, *Contemp. Math.* 267 (2000) 31–54.
- [17] Y. Doi, On the structure of relative Hopf modules, *Comm. Algebra* 11 (3) (1983) 243–255.
- [18] Y. Doi, Unifying Hopf modules, *J. Algebra* 153 (2) (1992) 373–385.
- [19] Y. Doi, M. Takeuchi, Cleft comodule algebras for a bialgebra, *Comm. Algebra* 14 (1986) 801–817.
- [20] J.M. Fernández Vilaboa, A.B. Rodríguez Raposo, Entwined modules and (co)cleft extensions, preprint, 2004.
- [21] A. Joyal, R. Street, Braided monoidal categories, *Adv. Math.* 102 (1993) 20–78.
- [22] M. Koppinen, Variations in the smash product with applications to group-graded rings, *J. Pure Appl. Algebra* 104 (1) (1995) 61–80.
- [23] H.F. Kreimer, M. Takeuchi, Hopf algebras and Galois extensions of an algebra, *Indiana Univ. Math. J.* 30 (1981) 675–691.
- [24] S. Majid, Cross products by braided groups and bosonization, *J. Algebra* 163 (1) (1994) 165–190.
- [25] D. Nikshych, L. de Vainerman, Finite quantum groupoids and their applications, in: *New Directions in Hopf Algebras*, in: MSRI Publ., vol. 43, 2002, pp. 211–262.
- [26] D. Nikshych, V. Turaev, L. de Vainerman, Invariants of knots and 3-manifolds from quantum groupoids, *Topology Appl.* 127 (2003) 91–123.
- [27] D.E. Radford, The structure of Hopf algebras with a projection, *J. Algebra* 92 (1985) 322–347.
- [28] D.E. Radford, J. Towber, Yetter–Drinfeld categories associated to an arbitrary bialgebra, *J. Pure Appl. Algebra* 87 (1993) 259–279.
- [29] D.N. Yetter, Quantum groups and representations of monoidal categories, *Math. Proc. Cambridge Philos. Soc.* 108 (2) (1990) 261–290.